

ON THE VERGNE CONJECTURE

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ABSTRACT. Consider a Hamiltonian action by a compact Lie group on a possibly non-compact symplectic manifold. We give a short proof of a geometric formula for decomposition into irreducible representations of the equivariant index of a Spin^c -Dirac operator in this context. This formula was conjectured by Michèle Vergne in 2006 and proved by Ma and Zhang in 2014.

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1. INTRODUCTION

Let us consider a Hamiltonian action by a compact, connected Lie group G , with Lie algebra \mathfrak{g} , on a possibly non-compact symplectic manifold (M, ω) with equivariant moment map $\mu : M \rightarrow \mathfrak{g}^*$. We assume that (M, ω) is *pre-quantisable*, that is, there exists a Hermitian line bundle E with a G -invariant Hermitian connection ∇^E such that

$$\frac{\sqrt{-1}}{2\pi}(\nabla^E)^2 = \omega. \quad (1.1)$$

We fix a G -invariant almost complex structure J on M so that

$$g^{TM}(X, Y) = \omega(X, JY), \quad X, Y \in TM, \quad (1.2)$$

defines a Riemmanian metric on M . The almost complex structure J determines a G -equivariant \mathbb{Z}_2 -graded spinor bundle

$$S_M^\pm = \wedge^{0,\text{even}/\text{odd}} T^*M,$$

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with *Clifford multiplication* denoted by

$$c : TM \rightarrow \text{End}(S_M).$$

Let ∇^{S_M} be a Hermitian Clifford connection on S_M , preserving S_M^\pm . One has the tensor product connection

$$\nabla := \nabla^{S_M} \otimes 1 + 1 \otimes \nabla^E$$

on $S_M \otimes E$. The associated Spin^c -Dirac operator D^E is defined by the following composition:

$$\Gamma(M, S_M \otimes E) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes S_M \otimes E) \xrightarrow{c} \Gamma(M, S_M \otimes E).$$

Here D^E anticommutes with the \mathbb{Z}_2 -grading, so that one has the operators

$$D_\pm^E : \Gamma(M, S_M^\pm \otimes E) \rightarrow \Gamma(M, S_M^\mp \otimes E).$$

By identifying $\mathfrak{g}^* \cong \mathfrak{g}$ via an Ad-invariant inner product, we also view μ as a map into \mathfrak{g} . This map induces a vector field X^μ on M via the infinitesimal action. Suppose the set $\{X^\mu = 0\}$ of zeroes of X^μ is compact. In Section 2, we review Braverman's equivariant index $\text{Ind}_G(M, \mu)$ of the deformed Dirac operator

$$D_\mu := D^E - \sqrt{-1} \cdot f \cdot c(X^\mu),$$

where $f \in C^\infty(M)^G$ satisfies certain growth conditions.

Let λ be the highest weight of an irreducible representation π_λ of G , for choices of a maximal torus and positive roots. Consider the reduced space $M_\lambda := \mu^{-1}(G \cdot \lambda)/G$. Suppose μ is proper. Then M_λ is compact, so one can define the index $\text{Ind}(M_\lambda) \in \mathbb{Z}$ of a Dirac operator on M_λ , and even make sense of this if λ is a singular value of μ . Vergne conjectured in her 2006 ICM plenary lecture [Ver07] that

$$\boxed{\text{Ind}_G(M, \mu)_\lambda = \text{Ind}(M_\lambda)}, \quad (1.3)$$

where we will always use a subscript λ to denote the multiplicity of π_λ . This is a generalisation of the quantisation commutes with reduction principle [GS82, Mei98, MS99, Par01, TZ98] to noncompact manifolds.

A special case of the Vergne conjecture, related to discrete series representations of semi-simple Lie groups, was studied by Paradan [Par03]. A generalisation of the Vergne conjecture, where the set $\{X^\mu = 0\}$ is not required to be compact, was first proved by Ma and Zhang [MZ14]. Later, Paradan gave a different proof [Par11]. This result was extended to Spin^c -manifolds in [HS15].

Our goal in the current paper is to give a short proof of the Vergne conjecture, Theorem 3.3. Many ideas we will use to prove the Vergne conjecture overlap with those used in the Spin^c setting in Section 5 of [HS15]. The reason the argument can be simplified in the symplectic case is that one has the equality (1.3) for $\lambda = 0$ to begin with (see Theorem 3.1), which is not true in the Spin^c case.

In [MZ14, Par11], it is not assumed that $\{X^\mu = 0\}$ is compact, just that μ is proper. This generalisation is natural, because one needs to allow noncompact vanishing sets for the crucial multiplicativity property of the index in [MZ14, Par11], even if $\{X^\mu = 0\}$ is compact for the initial moment map μ . We are able to avoid this issue, and work with compact vanishing sets, by only proving multiplicativity of the invariant part of the index, as in Section 6. This proof is based on Braverman's cobordism invariance, and allows us to keep the proof of the Vergne conjecture short.

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2. BRAVERMAN'S INDEX

Let $\phi : M \rightarrow \mathfrak{g}$ be an equivariant map (with respect to the adjoint action by G on \mathfrak{g}). It induces the vector field X^ϕ on M defined by

$$X^\phi(m) := \left. \frac{d}{dt} \right|_{t=0} \exp(-t \cdot \phi(m)) \cdot m,$$

for $m \in M$.

Definition 2.1. The map ϕ is *taming* if the vanishing set $\{X^\phi = 0\} \subseteq M$ is compact.

Consider the deformed Dirac operator

$$D_\phi = D^E - \sqrt{-1} \cdot f \cdot c(X^\phi), \tag{2.1}$$

where $f : M \rightarrow [0, \infty[$ is a G -invariant smooth function which grows fast enough towards infinity. This is a so-called admissible function introduced by Braverman; for the precise growth condition see Definition 2.6 in [Bra02]. In [Bra02], Braverman defined an equivariant index of the deformed Dirac operator (2.1), for taming maps ϕ , in a more general context. He proved a cobordism invariance property of this index, which is the crucial ingredient of the arguments in this paper.

Braverman considered *complete* Riemannian manifolds. The manifolds we will consider (such as open subsets of a given manifold) may not be complete a priori. In such cases (possibly assuming the boundary of the manifold to be regular enough) one can make the manifold complete by rescaling the Riemannian metric by a positive, G -invariant function on M , which equals one in a neighbourhood of the zeroes of X^ϕ . (See Section 4.2 in [Bra02] for more details.) The resulting index is independent of the function used to rescale the metric. When dealing with non-complete manifolds in the context of Braverman's index, we will always tacitly perform this rescaling, and choose open sets so that this is possible. After rescaling, the Riemannian metric is only given by the symplectic form and the almost

complex structure as in (1.2) in a neighbourhood of the zeroes of X^ϕ , but that is enough for our arguments.

The results from [Bra02] that we will use are summarised in the following theorem. We will write $\widehat{R}(G) = \text{Hom}_{\mathbb{Z}}(R(G), \mathbb{Z})$ for the completion of the representation ring of G . Let T be a maximal torus of G , and \mathfrak{t} its Lie algebra. Let $\Lambda_+^* \subset \mathfrak{t}^*$ be the set of dominant weights, for a choice of positive roots. For $\lambda \in \Lambda_+^*$, let π_λ be the irreducible representation of G with highest weight λ . We will write D_ϕ^\pm for the restriction of D_ϕ to $\Gamma^\infty(S_M^\pm \otimes E)$.

Theorem 2.2 ([Bra02]). *If the map ϕ is taming, then the deformed Dirac operator D_ϕ has the following properties.*

- (1) *The kernel of D_ϕ decomposes, as a unitary representation of G , into an infinite direct sum*

$$\ker D_\phi^\pm = \bigoplus_{\lambda \in \Lambda_+^*} m_\lambda^\pm \cdot \pi_\lambda,$$

where m_λ^\pm is a nonnegative integer for every λ .

- (2) *The index*

$$\text{Ind}_G(M, \phi) := \sum_{\lambda \in \Lambda_+^*} (m_\lambda^+ - m_\lambda^-) \cdot \pi_\lambda \in \widehat{R}(G)$$

is independent of the choices of the admissible function f , and the connection ∇^{S_M} .

- (3) *If U is a G -invariant open subset of M so that*

$$\{X^\phi = 0\} \subseteq U \subseteq M,$$

then

$$\text{Ind}_G(M, \phi) = \text{Ind}_G(U, \phi|_U) \in \widehat{R}(G).$$

- (4) *If $(\phi^t)_{t \in [0,1]} : M \times [0,1] \rightarrow \mathfrak{g}$ is a smooth family of equivariant maps, which is taming over $M \times [0,1]$, and constant in t on $M \times [0, \epsilon[$ and $M \times]1 - \epsilon, 1]$ for an $\epsilon > 0$, then*

$$\text{Ind}_G(M, \phi^0) = \text{Ind}_G(M, \phi^1).$$

Remark 2.3. To define a G -equivariant index $\text{Ind}_G(M, \phi)$ for a taming map ϕ , one can also use Atiyah's index of transversally elliptic symbols as in [Par01, Par11], or an APS-type index as in [MZ14]. They are all consistent, see Theorem 5.5 in [Bra02] and Theorem 1.5 in [MZ14].

In what follows, we will study Braverman's index for $\phi = \mu$, where we identify $\mathfrak{g} \cong \mathfrak{g}^*$ via a fixed Ad-invariant inner product.

3. THE MAIN RESULT: VERGNE'S CONJECTURE

Take $\lambda \in \Lambda_+^*$. By identifying $\mathfrak{t}^* \cong \sqrt{-1}\mathfrak{t}^*$ via multiplication by $\sqrt{-1}$, we view λ as an element of \mathfrak{t}^* . If λ is a regular value of the moment map μ , then one can construct the Marsden-Weinstein symplectic reduction $(M_\lambda, \omega_\lambda)$, with $M_\lambda = \mu^{-1}(G \cdot \lambda)/G$ being a compact symplectic orbifold provided that μ is proper. Moreover, the pre-quantum line bundle E as well as the almost complex structure induce pre-quantum line bundle E_λ and almost complex structure J_λ on the reduced space $(M_\lambda, \omega_\lambda)$. Hence, one can define the orbifold index [Kaw81] $\text{Ind}(M_\lambda) \in \mathbb{Z}$ of a Spin^c -Dirac operator on M_λ .

If λ is not a regular value of μ , then one can show that for generic $\epsilon \in \mathfrak{g}$ such that $\lambda + \epsilon \in \mu(M)$, this element $\lambda + \epsilon$ is a regular value. Furthermore, the integer $\text{Ind}(M_{\lambda+\epsilon})$ is independent of small enough ϵ . (See Theorem 2.5 in [MS99] or Theorem C in [Par01].) One then defines

$$\text{Ind}(M_\lambda) := \text{Ind}(M_{\lambda+\epsilon}) \in \mathbb{Z},$$

for an ϵ as above.

Theorem 3.1. *Let (M, ω) be a Hamiltonian G -space with pre-quantum line bundle E and taming moment map μ . If $0 \notin \mu(M)$, then*

$$\text{Ind}_G(M, \mu)_0 = 0.$$

If $0 \in \mu(M)$, then

$$\text{Ind}_G(M, \mu)_0 = \text{Ind}(M_0) \in \mathbb{Z}. \quad (3.1)$$

Proof. Theorem C in [Par01] is the quantisation commutes with reduction result in the compact case. However, Paradan's arguments in Section 7 of that paper also imply this statement for noncompact manifolds. See also Theorem 4.3 in [TZ99]. \square

Remark 3.2. When the manifold M is compact, the moment map μ is automatically taming and proper. Then Theorem 3.1 is the Guillemin–Steinberg conjecture, which was first proved by Meinrenken [Mei98] and Meinrenken–Sjamaar [MS99]. Later, Tian–Zhang [TZ98] and Paradan [Par01] gave different proofs.

Michèle Vergne conjectured in her 2006 ICM plenary lecture [Ver07] that the identity (3.1) holds not only for the trivial representation but for all irreducible G -representations.

Theorem 3.3. *Let (M, ω) be a Hamiltonian G -space with pre-quantum line bundle E and proper, taming moment map μ . One has*

$$\text{Ind}_G(M, \mu) = \sum_{\lambda \in \Lambda_+^* \cap \mu(M)} \text{Ind}(M_\lambda) \pi_\lambda.$$

One can view $\text{Ind}_G(M, \mu)$ as the geometric quantisation of the action by G on (M, ω) . Vergne's conjecture then states that quantisation commutes with reduction in this context.

In the remainder of this paper, we give a proof of Theorem 3.3.

4. MAKING A TAMING MAP PROPER

For any $\xi \in \mathfrak{g}^* \cong \mathfrak{g}$, let ξ^M be the vector field induced by the infinitesimal action of G on M . Let $\mu_\xi \in C^\infty(M)$ be the pairing of μ with ξ . One has the Kostant formula

$$2\pi\sqrt{-1}\mu_\xi = \nabla_{\xi^M}^E - \mathcal{L}_\xi^E, \quad (4.1)$$

for $\xi \in \mathfrak{g}$, where \mathcal{L}_ξ^E is the Lie derivative of sections of E . If we choose a different G -invariant connection $\tilde{\nabla}^E$ on the pre-quantum line bundle E , we obtain a different map $\tilde{\mu} : M \rightarrow \mathfrak{g}^*$. We will still call such a map a moment map.

Lemma 4.1. *Let $\tilde{\mu}$ be an arbitrary moment map defined as in (4.1). Let $H \subset G$ be a closed subgroup with \mathfrak{h} its Lie algebra. If Z is a connected component of $M^H \cap \tilde{\mu}^{-1}(\mathfrak{h})$, then $\tilde{\mu}$ is constant over Z . In particular, $\tilde{\mu}(Z) \in \mathfrak{h}^*$ is given by the weight of the action of H on the line bundle E over Z .*

Proof. For any $\xi \in \mathfrak{h}$ and $m \in M^H$, we have that $\xi^M(m) = 0$. Thus, by (4.1),

$$\mu_\xi(m) = \frac{\sqrt{-1}}{2\pi} \cdot \mathcal{L}_\xi^{E_m},$$

which is determined by the weight of the action of H on E_m and is locally constant. \square

Let $U \subset M$ be a G -invariant open, relatively compact subset of M such that μ is taming over U . Let $(X^\mu)^*$ be the dual of the vector field X^μ , which is a G -invariant 1-form on U . For any G -invariant function χ on U , setting

$$\nabla_\chi^E = \nabla^E + 2\pi\sqrt{-1}\chi \cdot (X^\mu)^*$$

defines a new connection ∇_χ^E on E . Let $\mu_\chi : U \rightarrow \mathfrak{g}^*$ be the moment map determined by ∇_χ^E and (4.1). The following proposition plays a key role.

Proposition 4.2. *Let V be a G -invariant, relatively compact neighbourhood of $\{X^\mu = 0\} \cap U$ such that $\bar{V} \subset U$. We can choose the function χ so that*

- (1) μ_χ is proper;
- (2) $\mu_\chi|_V = \mu|_V$;
- (3) $\|\mu_\chi\| \geq \|\mu|_U\|$;
- (4) the vector fields $X^\mu|_U$ and X^{μ_χ} have the same set of zeroes.

Proof. Let $\{\xi_1, \dots, \xi_{\dim \mathfrak{g}}\}$ be an orthonormal basis of \mathfrak{g} . We define a map $\psi : U \rightarrow \mathfrak{g}$ by

$$\psi(m) := \sum_{j=1}^{\dim \mathfrak{g}} \langle X^\mu(m), \xi_j^M(m) \rangle \cdot \xi_j \in \mathfrak{g},$$

for $m \in M$. Then we have

$$\langle \psi, \mu \rangle = \|X^\mu\|^2 \quad \text{and} \quad \mu_\chi = \mu + \chi \cdot \psi.$$

The two maps $\mu, \psi : U \rightarrow \mathfrak{g}$ are bounded since U is relatively compact. Moreover, the assumption that $X^\mu \neq 0$ outside V ensures that there exists $\epsilon > 0$ so that $\|\psi\| > \epsilon$ over $U \setminus V$. Thus, μ_χ is proper as long as the function χ is a proper function over U .

If we choose the function χ so that $\chi \equiv 0$ on V , the second condition is satisfied. The third condition follows directly from the following inequality

$$\|\mu_\chi\| \cdot \|\mu\| \geq \langle \mu_\chi, \mu \rangle = \|\mu\|^2 + \chi \cdot \|X^\mu\|^2 \geq \|\mu\|^2.$$

It remains to compare the vanishing set of the vector fields X^μ and X^{μ_χ} . First, suppose $X^\mu(m) = 0$. Then $\mu_\chi(m) = \mu(m)$, so $X^{\mu_\chi}(m) = X^\mu(m) = 0$. To prove the converse implication, note that

$$\langle X^\mu, X^{\mu_\chi} \rangle = \|X^\mu\|^2 + \chi \cdot \sum_{j=1}^{\dim \mathfrak{g}} \langle X^\mu, \xi_j^M \rangle^2. \quad (4.2)$$

The second term on the right-hand side of the above equation is non-negative provided the function χ is non-negative. Then $X^{\mu_\chi}(m) = 0$ implies that $X^\mu(m) = 0$. This completes the proof. \square

5. A LOCALISATION ON PRODUCT MANIFOLDS

Suppose that N is a compact Hamiltonian G -space with pre-quantum line bundle F , and moment map $\mu^F : N \rightarrow \mathfrak{g}^*$. From now on, we will denote the moment map μ by μ^E , to make the distinction with μ^F clear. For any map $\psi : M \rightarrow \mathfrak{g}^*$, we abuse notation by also denoting the map $M \times N \rightarrow \mathfrak{g}^*$, mapping $(m, n) \in M \times N$ to $\psi(m)$, by ψ . (And similarly for maps from N to \mathfrak{g}^* .)

Let $U \subseteq M$ be a G -invariant open, relatively compact subset such that μ^E is taming over U . Fix a subset $V \subset U$ and a function χ as in Proposition 4.2. Let $\eta \in C^\infty(\mathbb{R})$ be a function with values in $[0, 1]$, and such that

$$\eta(t) = \begin{cases} 0 & \text{if } t \leq 1/3; \\ 1 & \text{if } t \geq 2/3. \end{cases}$$

Set $W := U \times N \times [0, 1]$, and consider the map $\phi : W \rightarrow \mathfrak{g}^*$ given by

$$\phi(m, n, t) = \mu_\chi^E(m) + \eta(t)\mu^F(n),$$

for $(m, n, t) \in W$. Since μ_χ^E is proper and μ^F is bounded, the map ϕ has to be proper.

Lemma 5.1. *The map ϕ is taming.*

Proof. The vanishing set of X^ϕ decomposes as

$$\{X^\phi = 0\} = \bigcup_H G \cdot (W^H \cap \phi^{-1}(\mathfrak{h})),$$

where H runs over the stabiliser groups of the action by T on W . Since U is relatively compact in M , only finitely many such stabilisers occur. Hence it is enough to prove that for each stabiliser H , the set

$$W^H \cap \phi^{-1}(\mathfrak{h}) \quad (5.1)$$

is compact.

Fix a stabiliser group H of the T -action on W and a connected component Z of $W^H \cap \phi^{-1}(\mathfrak{h})$. Suppose that $\alpha_Z, \beta_Z \in \mathfrak{h}^*$ are the weights of the action of H on the line bundles E and F restricted to Z . Then we have that

$$\phi(Z) \subset \alpha_Z + [0, 1]\beta_Z \subset \mathfrak{h}^*,$$

which is a compact segment. Since the closure $\overline{W^H}$ of W^H in $M \times N \times [0, 1]$ is compact, it has finitely many connected components. The weight of the action by H on a line bundle over $\overline{W^H}$ is constant on these connected components. So there are only finitely many elements $\alpha_Z, \beta_Z \in \mathfrak{h}^*$ as above (where H is fixed but Z may vary). Hence the set

$$\phi(W^H \cap \phi^{-1}(\mathfrak{h})) \quad (5.2)$$

is compact.

The point of using the proper moment map μ_χ^E rather than the original map μ^E is that this makes the map ϕ proper. Therefore, compactness of the set (5.2) implies compactness of the set (5.1). \square

A particular consequence of Lemma 5.1 is that the map

$$\mu_\chi^E + \mu^F = \phi(-, -, 1)$$

is taming. So the index

$$\text{Ind}_G(U_M \times N, \mu_\chi^E + \mu^F)$$

is well-defined. By Lemma 5.1 and the fourth point of Theorem 2.2, it equals

$$\text{Ind}_G(U_M \times N, \mu_\chi^E).$$

By Proposition 4.2, the vector fields induced by μ_χ^E and μ^E have the same set of zeroes, and are equal in a neighbourhood of that set. So by the third point of Theorem 2.2, we find that

$$\text{Ind}_G(U_M \times N, \mu_\chi^E + \mu^F) = \text{Ind}_G(U_M \times N, \mu^E) \in \widehat{R}(G). \quad (5.3)$$

6. PROOF OF THE VERGNE CONJECTURE

Let us fix a $\lambda \in \Lambda_+^*$, and let $N := G \cdot \lambda$ be the orbit through λ of the coadjoint action by G on \mathfrak{g}^* . Let F be the dual of the canonical pre-quantum holomorphic line bundle on N , so that the associated moment map μ^F is minus the inclusion $N \hookrightarrow \mathfrak{g}^*$. By the Borel–Weil–Bott theorem, we know that

$$\text{Ind}_G(N, F) = \pi_\lambda^*.$$

Let $R > \|\lambda\|^2$ be a regular value of the function $\|\mu^E + \mu^F\|^2 : M \times N \rightarrow \mathbb{R}$. Define

$$U_{M \times N} = \{(m, n) \in M \times N \mid \|\mu^E(m) + \mu^F(n)\|^2 < R\} \subseteq M \times N.$$

Then $U_{M \times N}$ is a G -invariant, open, relatively compact subset of $M \times N$. For a generic choice of R , the map $\mu^E + \mu^F$ is taming over $U_{M \times N}$, as we will assume. By the choice of N ,

$$(\mu^E + \mu^F)^{-1}(0) \cong (\mu^E)^{-1}(G \cdot \lambda).$$

By Theorem 3.1, we therefore have

$$\text{Ind}_G(U_{M \times N}, \mu^E + \mu^F)_0 = \text{Ind}(M_\lambda) \quad \in \mathbb{Z}, \quad (6.1)$$

if $\lambda \in \mu^E(M)$, and zero otherwise.

Choose $R' > 0$ large enough so that the set

$$U_M := \{m \in M \mid \|\mu^E(m)\|^2 < R'\}$$

contains $\{X^{\mu^E} = 0\}$. Again, we can choose R' such that μ^E is taming on U_M . In addition, choose $R' > R$ so that there is a G -invariant neighbourhood V_M of $\{X^{\mu^E} = 0\}$ such that $\overline{V_M} \subset U_M$, and

$$\overline{U_{M \times N}} \subset V_M \times N.$$

This is possible because μ^F is bounded on N . Let the function $\chi \in C^\infty(U_M)^G$ be as in Proposition 4.2, applied with $U = U_M$ and $V = V_M$. By (5.3), we have that

$$\text{Ind}_G(U_M \times N, \mu_\chi^E + \mu^F) = \text{Ind}_G(U_M \times N, \mu^E) \quad \in \widehat{R}(G).$$

In particular,

$$\text{Ind}_G(U_M \times N, \mu_\chi^E + \mu^F)_0 = (\text{Ind}_G(U_M, \mu^E) \otimes \pi_\lambda^*)_0 = \text{Ind}_G(M, \mu^E)_\lambda \quad \in \mathbb{Z}. \quad (6.2)$$

Because of (6.1) and (6.2), the last step in the proof of Theorem 3.3 is the following equality.

Lemma 6.1. *For R and R' large enough, one has*

$$\text{Ind}_G(U_M \times N, \mu_\chi^E + \mu^F)_0 = \text{Ind}_G(U_{M \times N}, \mu^E + \mu^F)_0 \quad \in \mathbb{Z}.$$

Proof. Corollary 6.18 in [Par01] (see also Theorem 9.6 in [PV15]) implies that, for R and R' large enough,

$$\text{Ind}_G(U_M \times N, \mu_\chi^E + \mu^F)_0 = \text{Ind}_G(U_{M \times N}, \mu_\chi^E + \mu^F)_0.$$

This follows from the fact that $U_{M \times N}$ is a neighbourhood of the set of zeroes of $\mu_\chi^E + \mu^F$. Here we have used the equivalence of Braverman's index and the index defined by Paradan and Vergne (see Remark 2.3). Inside $U_{M \times N}$, the function χ equals zero. Hence

$$\text{Ind}_G(U_{M \times N}, \mu_\chi^E + \mu^F) = \text{Ind}_G(U_{M \times N}, \mu^E + \mu^F).$$

□

Remark 6.2. In the proof of Lemma 6.1, we used Corollary 6.18 from [Par01]. That result states that the invariant part of the index vanishes if the norm of the moment map has a large enough lower bound. This was generalised to multiplicities of arbitrary irreducible representations in Theorem 2.1 in [MZ14] and Theorem 2.9 in [Par11] (in the symplectic setting) and Theorem 3.4 in [HS15] (in the Spin^c setting). Using one of the results in the symplectic setting, one can generalise the definition of the index in Theorem 2.2 to proper, non-taming moment maps (see Definition 1.3 in [MZ14] and Definition 2.10 in [Par11]). In addition, by using a suitable version of these vanishing results, one can generalise Lemma 6.1 to multiplicities of arbitrary irreducible representations. This can then be used to generalise Theorem 3.3 to proper, non-taming moment maps. Because our goal was to give a short proof of Vergne’s conjecture, we have not included the details of this generalisation in this paper.

REFERENCES

- [Bra02] Maxim Braverman. Index theorem for equivariant Dirac operators on noncompact manifolds. *K-Theory*, 27(1):61–101, 2002.
- [GS82] Victor Guillemin and Shlomo Sternberg. Geometric quantization and multiplicities of group representations. *Invent. Math.*, 67(3):515–538, 1982.
- [HS15] Peter Hochs and Yanli Song. Equivariant indices of Spin^c -dirac operators for proper moment maps. <http://arxiv.org/abs/1503.00801>, 2015.
- [Kaw81] Tetsuro Kawasaki. The index of elliptic operators over V -manifolds. *Nagoya Math. J.*, 84:135–157, 1981.
- [Mei98] Eckhard Meinrenken. Symplectic surgery and the Spin^c -Dirac operator. *Adv. Math.*, 134(2):240–277, 1998.
- [MS99] Eckhard Meinrenken and Reyer Sjamaar. Singular reduction and quantization. *Topology*, 38(4):699–762, 1999.
- [MZ14] Xiaonan Ma and Weiping Zhang. Geometric quantization for proper moment maps: the Vergne conjecture. *Acta Math.*, 212(1):11–57, 2014.
- [Par01] Paul-Émile Paradan. Localization of the Riemann-Roch character. *J. Funct. Anal.*, 187(2):442–509, 2001.
- [Par03] Paul-Émile Paradan. Spin^c -quantization and the K -multiplicities of the discrete series. *Ann. Sci. École Norm. Sup. (4)*, 36(5):805–845, 2003.
- [Par11] Paul-Émile Paradan. Formal geometric quantization II. *Pacific J. Math.*, 253(1):169–211, 2011.
- [PV15] Paul-Émile Paradan and Michèle Vergne. Witten non abelian localization for equivariant K -theory, and the $[Q, R] = 0$ theorem. <http://arxiv.org/abs/1504.07502v1>, 2015.
- [TZ98] Youliang Tian and Weiping Zhang. An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg. *Invent. Math.*, 132(2):229–259, 1998.
- [TZ99] Youliang Tian and Weiping Zhang. Quantization formula for symplectic manifolds with boundary. *Geom. Funct. Anal.*, 9(3):596–640, 1999.
- [Ver07] Michèle Vergne. Applications of equivariant cohomology. In *International Congress of Mathematicians. Vol. I*, pages 635–664. Eur. Math. Soc., Zürich, 2007.